Journal of Engineering Physics and Thermophysics, Vol. 74, No. 3, 2001

EFFECT OF LIQUID INERTIA AND WALL POROSITY ON THE PROCESS OF MOMENTUM TRANSFER IN A BIOBEARING WITH INJECTION OF A LUBRICANT

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UDC 621.89.001.57

We consider the flow of a compressed Newtonian fluid in a biological bearing that has a porous wall, i.e., in a hip joint. The system is modeled by two curvilinear surfaces and a porous wall bounded by a curved impermeable surface. The flow in the gap is considered with account for inertia forces. The Navier–Stokes and Poisson equations used are separated with the aid of the Morgan–Cameron approximation. A solution which relies on the averaging of inertia forces is obtained in a closed form. A bearing of spherical shape is considered as an example.

Introduction. Joints are biological bearings that connect the elements of human limbs. Depending on localization, they can be spherical, as in the case of hip joints, quasispherical for shoulder joints, or quasicylindrical for elbow and knee joints. These combinations are arranged like any other bearings. They consist of a pin and a coupling whose roles in a joint are played by the head of the bone and the socket-subheel. Both the "heel" and "subheel" of the joint are surrounded by cartilage, which is a porous structure having antifriction properties. On the surface of the heel the cartilage is more compact in structure than on the surface of the subheel, and in the first approximation it can be represented as an impermeable wall. The subheel is less dense in structure, and due to its larger thickness it can be modeled as a porous permeable layer. Biobearings are lubricated with synovial fluid, which has non-Newtonian properties. Its non-Newtonian behavior owes its origin not to the abnormal viscosity but rather to the difference of normal stresses being different from zero.

The aim of the present work is to give a mathematical description of the process of momentum transfer in biobearings. The hip joint is one of the most loaded joints in the human body. Hence, the proposed analysis may turn out to be useful for contemporary orthopedics.

In the work, the following assumptions were made: (a) the surfaces of the bearing are bodies of revolution with a common symmetry axis; (b) the synovial fluid is Newtonian; (c) dynamic loading of the bearing produces the effect of punching of the fluid film and leads to the necessity of taking into account inertia effects.

The above-considered problem in a biobearing with a permeable wall is analyzed with the aid of a unique method of averaging inertia forces [7–9].

Equations of Motion in the Gap of a Bearing. Let us consider a radial bearing with a working surface of curvilinear profile (Fig. 1). The geometry of the upper boundary of the porous layer is described by the function $R(x)$ of the boundary radius. The fluid film thickness is expressed by the function $h(x, t)$, and

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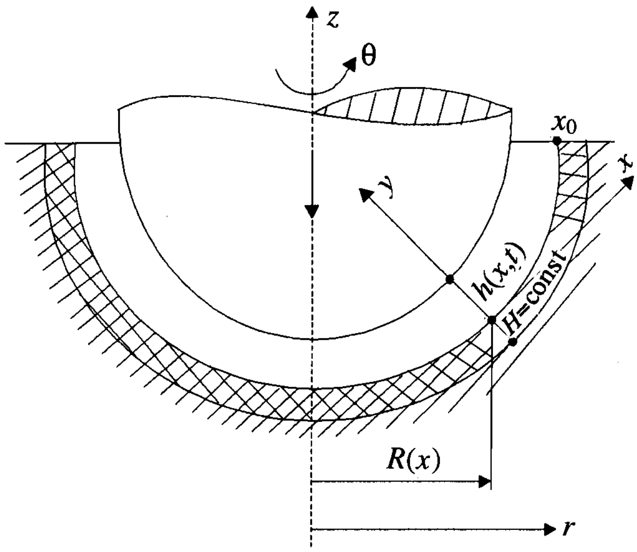


Fig. 1. Configuration of the radial bearing with a porous wall.

the thickness of the porous layer is constant, H = const. The corresponding curvilinear orthogonal coordinate system (x, θ, y) is also shown in Fig. 1.

Using the assumptions of hydrodynamic lubrication, the equations of motion of the Newtonian fluid, with account for the axial symmetry, can be presented in the form [7–9]

$$\frac{1}{R}\frac{\partial (R v_x)}{\partial x} + \frac{\partial v_y}{\partial y} = 0 , \tag{1}$$

$$\rho \left[\frac{\partial v_x}{\partial t} + \left(\frac{R'}{R} + \frac{\partial}{\partial x} \right) v_x^2 + \frac{\partial}{\partial y} (v_x v_y) \right] = - \frac{dp}{dx} + \mu \frac{\partial^2 v_x}{\partial y^2} . \tag{2}$$

The boundary-value conditions of the problem are

$$v_x (x, 0, t) = 0 , \quad v_x (x, h, t) = 0 ; \tag{3}$$

$$v_y (x, 0, t) = V_H , \quad v_y (x, h, t) = \frac{\partial h}{\partial t} = \dot{h} ; \tag{4}$$

$$\left. \frac{dp}{dx} \right|_{x=0} = 0 , \quad p (x_0) = p_0 . \tag{5}$$

$$\frac{\partial^2 v_x}{\partial y^2} = f (x, t) , \tag{6}$$

where

$$f (x, t) = \frac{1}{\mu} \frac{dp}{dx} + \frac{\rho}{\mu h} \frac{\partial}{\partial t} \int_0^h v_x dy + \frac{\rho}{\mu h} \left(\frac{R'}{R} + \frac{\partial}{\partial x} \right) \int_0^h v_x^2 dy . \tag{7}$$

The solution of Eq. (6) has the form

$$v_x = \frac{f}{2}\,(y^2 - yh)\ . \tag{8}$$

The substitution of Eq. (8) into Eq. (1) leads to a modified Reynolds equation:

$$\frac{1}{R}\frac{\partial}{\partial x}\,(Rh^3 f) = 12\left(\frac{\partial h}{\partial t} - V_H\right), \tag{9}$$

in which the function f is equivalent to the pressure gradient.

For the solution of Eq. (9) we consider the flow of liquid in a porous layer. According to the Darcy equations, the flow velocity components are

$$V_x = -\frac{\Phi}{\mu}\frac{\partial P}{\partial x}\,, \quad V_y = -\frac{\Phi}{\mu}\frac{\partial P}{\partial y}\ . \tag{10}$$

The transverse velocity component must be continuous on the interface between the porous wall and the liquid film and equal to V_H. Consequently, with account for Eqs. (4) and (10) we have

$$\frac{1}{R}\frac{\partial}{\partial x}\,(Rh^3 f) = 12\left[\frac{\partial h}{\partial t} + \frac{\Phi}{\mu}\left(\frac{\partial P}{\partial y}\right)_{y=0}\right]. \tag{11}$$

On substitution of Eq. (10) into the continuity equation written similarly to Eq. (1) for the liquid in the porous wall, we obtain the Poisson equation

$$\frac{1}{R}\frac{\partial}{\partial x}\,R\left(\frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial y}\right) = 0 \tag{12}$$

and using the Morgan–Cameron approximation [10], we find

$$\left(\frac{\partial P}{\partial y}\right)_{y=0} = \frac{\mu H}{R}\frac{\partial}{\partial x}\,(Rf)\ . \tag{13}$$

The substitution of Eq. (13) into Eq. (11) yields the modified Reynolds equation

$$\frac{1}{R}\frac{\partial}{\partial x}\,[R\,(h^3 + 12\,\Phi H)\,f] = 12\,\frac{\partial h}{\partial t}\ . \tag{14}$$

Its solution is

$$f = \frac{12}{R\,(h^3 + 12\,\Phi H)}\int R\dot{h}dx\ . \tag{15}$$

From Eq. (8), with Eq. (15) taken into account, we find the differential equation for the derivative of the pressure dp/dx. Its integration yields

$$p\,(x, t) = p_0 - 12\mu\,[S_0 - S\,(x, t)] - \rho\,[I_0 - I\,(x, t)] + \frac{6\rho}{5}\,[T_0 - T\,(x, t)]\ , \tag{16}$$

where

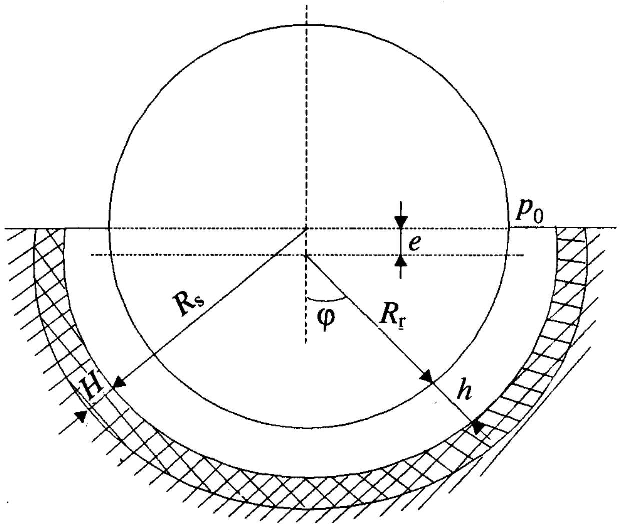


Fig. 2. Configuration of the spherical biobearing.

$$A(x,t)=\int R\dot{h}dx\,,\quad B(x,t)=\frac{A(x,t)}{R(h^3+12\,\Phi H)}\,,\quad C(x,t)=\frac{h^3A(x,t)}{R(h^3+12\,\Phi H)}\,,$$

$$S(x,t)=\int B(x,t)\,dx\,,\quad I(x,t)=\int\frac{1}{h}\frac{\partial C(x,t)}{\partial t}dx\,,\quad T(x,t)=\int\frac{1}{h}\left(\frac{R'}{R}+\frac{\partial}{\partial x}\right)\frac{C^2(x,t)}{h}dx\,. \tag{17}$$

The load-carrying capacity of the bearing is described by the expression

$$N=2\pi\int_0^{x_0}pR\cos\varphi\,dx\,. \tag{18}$$

Application of the "Spherical Bearing" Method (Fig. 2). Let us introduce the following dimensionless parameters:

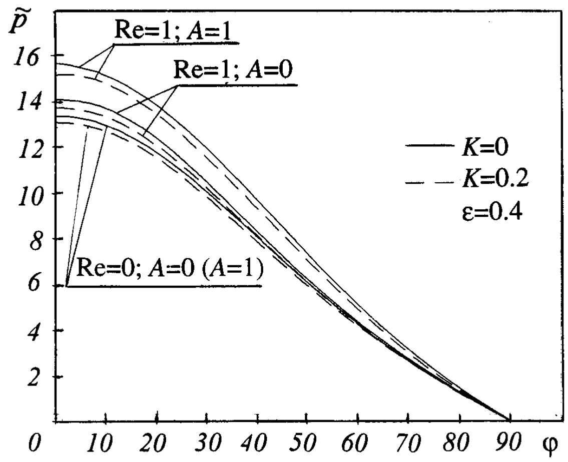
$$C=R_s-R_r\,,\quad \varepsilon=\frac{e}{C}\,,\quad \dot{\varepsilon}=\frac{d\varepsilon}{dt}\,,\quad \varphi=\frac{x}{R_r}\,,$$

$$\Psi=\frac{h}{C}=1-\varepsilon\cos\varphi\,,\quad E=1-\varepsilon\,,\quad \dot{h}=-C\dot{\varepsilon}\cos\varphi\,,\quad R=R_r\sin\varphi\,, \tag{19}$$

$$A=\frac{\ddot{\varepsilon}\varepsilon}{\dot{\varepsilon}^2}\,,\quad K=\frac{12\Phi H}{C^3}\,,\quad \mathrm{Re}=\frac{\rho C^2\dot{\varepsilon}}{\mu}\,.$$

We avail ourselves of the relations for pressure distribution in the gap of the bearing and for the load-carrying capacity of the latter (assuming that $K<1$ and ignoring the terms that contain the parameter K raised to a power greater than three):

$$\tilde{p}=\frac{(p-p_0)\,C^2}{\mu R_r^2\dot{\varepsilon}}=\frac{3}{\varepsilon}\left\{\frac{1}{\Psi^2}-1-\frac{2K^3}{5}\left[\frac{1}{\Psi^5}-1\right]\right\}+F_p(\mathrm{Re},A,K,\varphi)\,, \tag{20}$$



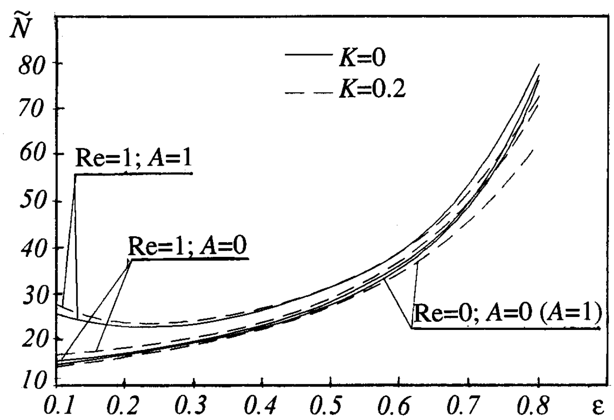


Fig. 3. Pressure distribution in the gap of the spherical biobearing.

Fig. 4. Load-carrying capacity of the spherical biobearing.

$$\tilde{N} = \frac{NC^2}{\mu R_r^4 \dot{\varepsilon}} = \frac{6\pi}{\varepsilon^3}\left\{\frac{\varepsilon}{E} - \frac{\varepsilon^2}{2} + \ln E - \frac{K^3}{30}\left[1 - 6\varepsilon^2 - \frac{1-4\varepsilon}{E^4}\right]\right\} + F_N\,(\mathrm{Re}, A, K, \varepsilon)\,, \tag{21}$$

in which F_p and F_N are given by the relations

$$F_p = \frac{\mathrm{Re}\,A}{2\varepsilon^2}\ln\Psi - \frac{K^3\,\mathrm{Re}\,A}{6\varepsilon^2}\left(\frac{1}{\Psi^2} - 1\right) - \frac{K^3\,\mathrm{Re}}{8\varepsilon^2}\left(\frac{3-4\Psi}{\Psi^4} + 1\right) + \frac{\mathrm{Re}}{10\varepsilon^2}\left\{\left(\frac{1}{\Psi} - 1\right) + \right.$$
$$\left. + \frac{1-\varepsilon^2}{2}\left(\frac{1}{\Psi^2} - 1\right) + 2\ln\Psi - 2K^3\left[\left(\frac{1}{\Psi^3} - 1\right) - \frac{5}{4}\left(\frac{1}{\Psi^4} - 1\right) + \frac{4\,(1-\varepsilon^2)}{5}\left(\frac{1}{\Psi^5} - 1\right)\right]\right\}, \tag{22}$$

$$F_N = \frac{\pi\,\mathrm{Re}\,A}{4\varepsilon^4}\,[2\varepsilon + \varepsilon^2 + 2\,(1+\varepsilon^2)\ln E] - \frac{\pi K^3\,\mathrm{Re}\,A}{6\varepsilon}\,\frac{2-\varepsilon}{E^2} - \frac{\pi K^3\,\mathrm{Re}\,A}{8}\,\frac{3-\varepsilon}{E^3} +$$
$$+ \frac{3\pi\,\mathrm{Re}}{10\varepsilon^2}\left\{-\frac{2}{\varepsilon^2}\left(\ln E + \frac{2\varepsilon+\varepsilon^2}{2}\right) + \frac{1-\varepsilon^2}{\varepsilon}\left(\frac{\varepsilon}{E} - \frac{\varepsilon^2}{2} + \ln E\right) - \frac{1}{\varepsilon^2}\,[2\varepsilon + \varepsilon^2 + 2\,(1-\varepsilon^2)\ln E] - \right.$$
$$\left. - 2K^3\left[\frac{2\varepsilon-\varepsilon^2}{E^2} - \frac{5}{4}\left(\frac{3-\varepsilon}{3} - 1\right) + \frac{2}{15}\,\frac{1-\varepsilon^2}{\varepsilon^2}\left(1 - 6\varepsilon^2 - \frac{1-4\varepsilon}{E^4}\right)\right]\right\}. \tag{23}$$

The pressure distribution and the load-carrying capacity of the bearing are shown in Figs. 3 and 4.

Conclusions. The specific case of a spherical biobearing formulated on the basis of the foregoing general propositions allow the following conclusions:

(a) account for inertia effects in the acceleration of the punched film ($\mathrm{Re} \neq 0$, $A \neq 0$) leads to a considerable increase in the modeled pressure in comparison with the case where inertia forces are disregarded: ($\mathrm{Re} = 0$, $A = 0$);

(b) when the permeability of the wall is small ($K < 1$), the pressure turns out to be much lower than in the case of an impermeable wall;

(c) the change in pressure entails a corresponding change in the load-carrying capacity of the biobearing.

NOTATION

$R(x)$, radius of the upper boundary of the porous layer; H = const, thickness of the porous layer; V_x and V_y, components of the fluid velocity vector in the porous layer; V_H, transverse component of velocity on the upper boundary of the porous layer; R_s, radius of the coupling; R_r, radius of the pin; P, hydrodynamic pressure in a porous layer; p, hydrodynamic pressure in the gap; $h(x, t)$, fluid film thickness; $h = \partial h/\partial t$; v_x and v_y, components of the vector of the fluid velocity in the gap; e, distance between the centers of the circles that form the surfaces of the pin and coupling of the bearing; Φ, permeability of the porous layer; ρ, fluid density; μ, fluid viscosity; $\varphi = \xi/R$, angular measure of the curvilinear coordinate x. Subscripts: 0, the value of the parameter at the exit from the bearing. Superscript: prime, derivative with respect to x.

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